

# COMPUTING MODULAR EQUATIONS FOR SHIMURA CURVES

YIFAN YANG

**ABSTRACT.** In the classical setting, the modular equation of level  $N$  for the modular curve  $X_0(1)$  is the polynomial relation satisfied by  $j(\tau)$  and  $j(N\tau)$ , where  $j(\tau)$  is the standard elliptic  $j$ -function. In this paper, we will describe a method to compute modular equations in the setting of Shimura curves. The main ingredient is the explicit method for computing Hecke operators on the spaces of modular forms on Shimura curves developed in [15].

## 1. INTRODUCTION

Let  $j(\tau)$  be the elliptic  $j$ -function. For a positive integer  $N$ , consider the set

$$\begin{aligned}\Gamma_N &= \mathrm{SL}(2, \mathbb{Z}) \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix} \mathrm{SL}(2, \mathbb{Z}) \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = N, \gcd(a, b, c, d) = 1 \right\}.\end{aligned}$$

The group  $\mathrm{SL}(2, \mathbb{Z})$  acts on  $\Gamma_N$  by multiplication on the left and it can be easily checked that the cardinality of  $\mathrm{SL}(2, \mathbb{Z}) \backslash \Gamma_N$  is finite. Moreover, the multiplication of any element in  $\mathrm{SL}(2, \mathbb{Z})$  on the right of the cosets in  $\mathrm{SL}(2, \mathbb{Z}) \backslash \Gamma_N$  simply permutes the cosets. Thus, any symmetric sum of  $j(\gamma\tau)$ , where  $\gamma$  runs over a complete set of representatives for  $\mathrm{SL}(2, \mathbb{Z}) \backslash \Gamma_N$ , will be a modular function on  $\mathrm{SL}(2, \mathbb{Z})$  and therefore can be written as a rational function of  $j(\tau)$ . In fact, because the only possible pole occurs at the cusp, this rational function of  $j(\tau)$  is actually a polynomial. Then the *modular polynomial*  $\Phi_N(x, y)$  of level  $N$  is defined to be the polynomial in  $\mathbb{C}[x, y]$  such that

$$\Phi_N(x, j(\tau)) = \prod_{\gamma \in \mathrm{SL}(2, \mathbb{Z}) \backslash \Gamma_N} (x - j(\gamma\tau)).$$

In fact, it can be easily proved that this polynomial  $\Phi_N(x, y)$  is in  $\mathbb{Z}[x, y]$ . Since  $Y_0(1)$  is the moduli space of isomorphism classes of elliptic curves over  $\mathbb{C}$ , the equation  $\Phi_N(x, y) = 0$  is the *modular equation* of level  $N$  for moduli of elliptic curves over  $\mathbb{C}$ . That is, if  $E_1$  and  $E_2$  are two elliptic curves over  $\mathbb{C}$  admitting a cyclic  $N$ -isogeny between them, then their  $j$ -invariants  $j(E_1)$  and  $j(E_2)$  satisfy  $\Phi_N(j(E_1), j(E_2)) = 0$ .

Observe that if the Fourier expansion of  $j(\tau)$  is  $q^{-1} + 744 + 196884q + \cdots$ ,  $q = e^{2\pi i\tau}$ , then the Fourier expansion of  $j(N\tau)$  is simply  $q^{-N} + 744 + 196884q^N + \cdots$ . Thus, in principle, to determine  $\Phi_N(x, y)$ , one just has to compute enough Fourier coefficients for

---

*Date:* March 4, 2013.

*2000 Mathematics Subject Classification.* Primary 11F03 secondary 11F12, 11G18.

*Key words and phrases.* Hecke operators, modular equations, Schwarzian differential equations, Shimura curves.

The author would like to thank Professor John McKay and Professor John Voight for providing valuable comments on the paper. The author was partially supported by Grant 99-2115-M-009-011-MY3 of the National Science Council, Taiwan (R.O.C.).

$j(\tau)$  and solve a system of linear equations. Of course, the main difficulty in practice is that the coefficients are gigantic. On the other hand, because the coefficients of  $\Phi_N(x, y)$  are all integers, one can compute the reduction of  $\Phi_N(x, y)$  modulo  $p$  for a suitable number of primes  $p$  and then use the Chinese remainder theorem to recover the coefficients. See [3] for the current state of the art in the computation of  $\Phi_N(x, y)$ .

In this paper, we shall consider modular equations in the settings of Shimura curves. Let  $B$  be an indefinite quaternion algebra of discriminant  $D > 1$  over  $\mathbb{Q}$  and  $\mathcal{O}$  be a maximal order in  $B$ . Choose an embedding  $\iota : B \hookrightarrow M(2, \mathbb{R})$  and let

$$\Gamma(\mathcal{O}) := \{\iota(\alpha) : \alpha \in \mathcal{O}, N(\alpha) = 1\}$$

be the image of the norm-one group of  $\mathcal{O}$  under  $\iota$ , where  $N(\alpha)$  denotes the reduced norm of  $\alpha \in B$ . Then the Shimura curve  $X_0^D(1)$  is defined to be the quotient space  $\Gamma(\mathcal{O}) \backslash \mathbb{H}$ . As shown in [9], the Shimura curve  $X_0^D(1)$  is the coarse moduli space for isomorphism classes of abelian surfaces with quaternionic multiplication (QM) by  $\mathcal{O}$ . Let  $W_D$  denote the group of Atkin-Lehner involutions on  $X_0^D(1)$ . For our purpose, we will also consider the quotient curves of  $X_0^D(1)$  by subgroups  $W$  of  $W_D$ .

Now assume that  $X_0^D(1)$  has genus 0 and choose a Hauptmodul  $t(\tau)$  for  $X_0^D(1)$  so that  $t(\tau)$  generates the function field on  $X_0^D(1)$ . For a positive integer  $N$  relatively prime to  $D$ , pick an element  $\alpha$  of norm  $N$  in  $\mathcal{O}$  such that  $\mathcal{O} \cap (\alpha^{-1}\mathcal{O}\alpha)$  is an Eichler order of level  $N$ . Then the *modular polynomial* of level  $N$  for  $X_0^D(1)$  is defined to be the polynomial  $\Phi_N^D(x, y)$  of minimal degree, up to scalars, such that  $\Phi_N^D(t(\tau), t(\iota(\alpha)\tau)) = 0$ , which is essentially the rational function  $\tilde{\Phi}_N^D(x, y)$  such that

$$\tilde{\Phi}_N^D(x, t(\tau)) = \prod_{\gamma \in \Gamma(\mathcal{O}) \backslash \Gamma(\mathcal{O})\iota(\alpha)\Gamma(\mathcal{O})} (x - t(\gamma\tau)).$$

Here unlike the case of the modular curve  $X_0(1)$ , a symmetric sum of  $t(\gamma\tau)$  as  $\gamma$  runs through representatives of  $\Gamma(\mathcal{O}) \backslash \Gamma(\mathcal{O})\iota(\alpha)\Gamma(\mathcal{O})$  is not equal to a polynomial of  $t(\tau)$  in general. Since  $X_0^D(1)$  is the moduli space of abelian surfaces over  $\mathbb{C}$  with quaternionic multiplication, the modular equation of level  $N$  relates the moduli of two abelian surfaces with QM that have a certain type of isogenies between them. (The precise description of the isogeny is a little complicated to be given here. See [7, Appendix A] for details.) Modular equations for Atkin-Lehner quotients  $X_0^D(1)/W$  are similarly defined.

When  $D > 1$ , the problem of explicitly determining modular equations for Shimura curves is significantly more complicated than its classical counterpart. The reasons are that Shimura curves do not have cusps and it is difficult to find the Taylor expansion of an automorphic function with respect to a local parameter at any given point on the Shimura curve. When  $D$  and  $N$  are both small such that  $X_0^D(N)$  is also of genus 0, it is possible to work out an explicit cover  $X_0^D(N) \rightarrow X_0^D(1)$  from the ramification data alone. Then from the explicit cover, one can compute the modular equation of level  $N$ . This has been done in [4] for a limited number of cases.

Another possible method to compute modular equations for Shimura curves uses the Schwarzian differential equation associated to a Hauptmodul  $t(\tau)$ . (See Section 2 for a review on the notion and properties of Schwarzian differential equation.) The idea is that with a properly chosen pair of solutions  $F_1(t)$  and  $F_2(t)$  of the Schwarzian differential equation and a correct positive integer  $e$ , the expression  $(F_2(t)/F_1(t))^e$  can be taken to be a local parameter at the point  $\tau_0$  of the Shimura curve with  $t(\tau_0) = 0$ . Inverting the expression, one gets the Taylor expansion of  $t$  with respect to the local parameter. If somehow one manages to find the Taylor expansion of  $t(\iota(\alpha)\tau)$  (this can always be done when  $t(\iota(\alpha)\tau_0) = 0$ , then by computing enough terms and solving a system of linear

equations, one gets the modular equation. However, as far as we know, there does not seem to be any paper in the literature that employs this idea to obtain modular equations for Shimura curves. (The paper [1] did obtain the Taylor expansion for the Hauptmodul in the case  $D = 6$ . More recently, Voight and Willis [13] developed a method for numerically computing Taylor expansions of automorphic forms on Shimura curves.)

In his Ph.D. thesis [12], Voight developed a method to compute modular equations for Shimura curves associated to quaternion algebras over totally real number fields in the cases when the Shimura curves have genus zero and precisely three elliptic points, i.e., Shimura curves associated to arithmetic triangle groups. The idea is to use hypergeometric functions, which are essentially solutions of the Schwarzian differential equations mentioned above, to numerically determine coordinates of CM-points and then use the existence of canonical models and explicit Shimura reciprocity laws to determine modular equations. Even though the equations are obtained numerically, they can be verified rigorously by showing that the monodromy group of the branched cover is correct.

In this paper, we shall present a new method to compute modular equations for Shimura curves. Our method also uses Schwarzian differential equations, but relies more heavily on the arithmetic side of the theory. Namely, in an earlier work [15], we showed that the spaces of automorphic forms of any given weight on a Shimura curve  $X_0^D(1)$  of genus 0 can be completely characterized in terms of solutions of the Schwarzian differential equation. We [15] then devised a method to compute Hecke operators with respect to our basis. The Jacquet-Langlands correspondence plays a crucial role in our approach. It turns out that our method and results in [15] can also be used to compute modular equations for Shimura curves. We will describe the procedure in Section 3 and give a detailed example in Section 4. Using the computer algebra system MAGMA [2], we have succeeded in determining modular equations of prime level up to 19 for  $X_0^6(1)/W_6$  and those of prime level up to 23 for  $X_0^{10}(1)/W_{10}$ .

The main difficulty in generalizing our method to general Shimura curves lies at the fact that our method requires that a Schwarzian differential equation is known beforehand. Because of the problem of the existence of accessory parameters, the determination of Schwarzian differential equations usually requires that an explicit cover  $X_0^D(N) \rightarrow X_0^D(1)$  is known, which can be problematic when  $D$  is large. (In some sense, what we do here is to deduce modular equations of higher levels from that of a given small level.) Nonetheless, once an explicit cover of Shimura curves is determined, the combination of the methods in [15] and in this paper will yield modular equations for the Shimura curve.

The rest of the paper is organized as follows. In Section 2, we review the definition and properties of Schwarzian differential equations. In Section 3, we describe our method to compute modular equations for Shimura curves  $X_0^D(1)/W$ , assuming that an explicit cover  $X_0^D(p_0)/W \rightarrow X_0(1)/W$  is known. In Section 4, we give a detailed example illustrating our method. In Sections 5 and 6, we list part of our computational results for the Shimura curves  $X_0^6(1)/W_6$  and  $X_0^{10}(1)/W_{10}$ . Files in the MAGMA-readable format containing all our computational results are available upon request.

## 2. SCHWARZIAN DIFFERENTIAL EQUATIONS

In this section, we will review the definition and properties of Schwarzian differential equations. In particular, assuming the Shimura curve has genus 0, we will recall the characterization of the spaces of automorphic forms in terms of solutions of the associated Schwarzian differential equation. The method for computing Hecke operators on these

spaces is too complicated to describe here. We refer the reader to [15] for details. Most materials in this section are taken from [15].

Let  $X$  be a Shimura curve. We assume that the associated quaternion algebra is not  $M(2, \mathbb{Q})$  so that  $X$  has no cusps. Let  $F(\tau)$  be a (meromorphic) automorphic form of weight  $k$  and  $t$  be a non-constant automorphic function on  $X$ . It is known since the nineteenth century that the functions  $F, \tau F, \dots, \tau^k F$ , as functions of  $t$ , are solutions of a certain  $(k+1)$ -st linear ordinary differential equations with algebraic functions as coefficients. (See [11, 14].) In particular, since  $t'(\tau)$  is a meromorphic automorphic form of weight 2, the function  $t'(\tau)^{1/2}$  as a function of  $t$ , satisfies a second-order linear ordinary differential equation. We call this differential equation the *Schwarzian differential equation* associated to  $t$ , which has the following properties.

**Proposition 1** ([15, Proposition 6]). *Let  $X$  be a Shimura curve of genus zero with elliptic points  $\tau_1, \dots, \tau_r$  of order  $e_1, \dots, e_r$ , respectively. Let  $t(\tau)$  be a Hauptmodul of  $X$  and set  $a_i = t(\tau_i)$ ,  $i = 1, \dots, r$ . Then  $t'(\tau)^{1/2}$ , as a function of  $t$ , satisfies the differential equation*

$$\frac{d^2}{dt^2}F + Q(t)F = 0,$$

where

$$Q(t) = \frac{1}{4} \sum_{j=1, a_j \neq \infty}^r \frac{1 - 1/e_j^2}{(t - a_j)^2} + \sum_{j=1, a_j \neq \infty}^r \frac{B_j}{t - a_j}$$

for some constants  $B_j$ . Moreover, if  $a_j \neq \infty$  for all  $j$ , then the constants  $B_j$  satisfy

$$\sum_{j=1}^r B_j = \sum_{j=1}^r \left( a_j B_j + \frac{1}{4}(1 - 1/e_j^2) \right) = \sum_{j=1}^r \left( a_j^2 B_j + \frac{1}{2}a_j(1 - 1/e_j^2) \right) = 0.$$

Also, if  $a_r = \infty$ , then  $B_j$  satisfy

$$\sum_{j=1}^{r-1} B_j = 0, \quad \sum_{j=1}^{r-1} \left( a_j B_j + \frac{1}{4}(1 - 1/e_j^2) \right) = \frac{1}{4}(1 - 1/e_r^2).$$

We remark that when the Shimura curve has genus 0 and precisely 3 elliptic points, the relations among  $B_j$  are enough to determine the constants  $B_j$ . This reflects the fact in classical analysis that a second-order Fuchsian differential equation with exactly three singularities is completely determined by the local exponents. When the Shimura curve has more than 3 elliptic points, the relations are not enough to determine  $B_j$ . In literature, we refer to this kind of situations by saying that *accessory parameters* exist. In order to determine the accessory parameters, one usually tries to find an explicit cover of Shimura curves and use it to determine the Schwarzian differential equations associated to the two curves simultaneously.

Now one of the key observations in [15] is that the analytic behavior of the Hauptmodul  $t'(\tau)$  is very easy to determine and from this, one can work out a basis for the space of automorphic forms of even weight  $k$  in terms of  $t'(\tau)$ .

**Proposition 2** ([15, Theorem 4]). *Assume that a Shimura curve  $X$  has genus zero with elliptic points  $\tau_1, \dots, \tau_r$  of order  $e_1, \dots, e_r$ , respectively. Let  $t(\tau)$  be a Hauptmodul of  $X$  and set  $a_i = t(\tau_i)$ ,  $i = 1, \dots, r$ . For a positive even integer  $k \geq 4$ , let*

$$d_k = \dim S_k(X) = 1 - k + \sum_{j=1}^r \left\lfloor \frac{k}{2} \left( 1 - \frac{1}{e_j} \right) \right\rfloor.$$

Then a basis for the space of automorphic forms of weight  $k$  on  $X$  is

$$t'(\tau)^{k/2} t(\tau)^j \prod_{i=1, a_i \neq \infty}^r (t(\tau) - a_i)^{-\lfloor k(1-1/e_i)/2 \rfloor}, \quad j = 0, \dots, d_k - 1.$$

The combination of two propositions shows that all automorphic forms of a given even weight  $k$  can be expressed in terms of the solutions of the Schwarzian differential equation. In [15], the author developed a method for computing Hecke operators relative to the basis in Proposition 2. The key ingredients are the Jacquet-Langlands correspondence and explicit covers of Shimura curves. We refer the reader to [15] for details.

### 3. COMPUTING MODULAR EQUATIONS FOR SHIMURA CURVES

In this section, we will present a method for computing modular equations for Shimura curves, under the working assumptions that the Schwarzian differential equations have been determined and, for a certain prime fixed  $p_0$ , the matrices for the Hecke operator  $T_{p_0}$  with respect to the bases in Proposition 2 have already been computed for sufficiently many  $k$  according to the recipe in [15].

Let  $X$  be a Shimura curve of genus 0 of the form  $X_0^D(1)/W$  for some subgroup  $W$  of the group of Atkin-Lehner involutions and  $\Gamma$  be the discrete subgroup of  $\mathrm{SL}(2, \mathbb{R})$  corresponding to  $X$ . For a prime  $p$  not dividing  $D$ , let  $\gamma_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in \mathrm{GL}^+(2, \mathbb{R})$ ,  $j = 0, \dots, p$ , be representatives for the cosets defining the Hecke operator  $T_p$ , that is,

$$(1) \quad T_p : f \rightarrow p^{k/2-1} \sum_{j=0}^p \frac{(\det \gamma_j)^{k/2}}{(c_j \tau + d_j)^k} f(\gamma_j \tau).$$

Then the modular equation  $\Phi_p(x, y) = 0$  of level  $p$  is the polynomial relation between a Hauptmodul  $t(\tau)$  for  $X$  and  $t(\gamma_j \tau)$  for arbitrary  $j$ .

Pick any nonzero automorphic form  $F(\tau)$  on  $X$  with the smallest possible weight  $k$ . For convenience, we let

$$(2) \quad F_j(\tau) = \frac{(\det \gamma_j)^{k/2}}{(c_j \tau + d_j)^k} F(\gamma_j \tau),$$

the summands in (1). Then any symmetric sum of  $F_j(\tau)/F(\tau)$ ,  $j = 0, \dots, p$ , will be an automorphic function on  $X$  and hence can be expressed as a rational function of  $t(\tau)$ . In other words, there is a rational function  $\Psi(z, x)$  such that

$$(3) \quad \Psi(z, t(\tau)) = \prod_{j=0}^p \left( z - \frac{F_j(\tau)}{F(\tau)} \right).$$

That is, we have

$$(4) \quad \Psi \left( \frac{F_j(\tau)}{F(\tau)}, t(\tau) \right) = 0$$

for all  $j$ .

**Lemma 3.** *Let  $\Psi(z, x)$  be the rational function defined by (3). Let  $\Psi'(z, y)$  be the rational function such that*

$$\Psi'(z, y) = z^{p+1} \Psi(1/z, y),$$

*which is a polynomial in  $z$ . Let  $R(x, y)$  be the resultant of  $\Psi(z, x)$  and  $\Psi'(z, y)$  with respect to the variable  $z$ . Then the modular polynomial  $\Phi_p(x, y)$  appears as one of the irreducible factors over  $\mathbb{C}$  in the numerator of the rational function  $R(x, y)$ .*

*Proof.* Let  $\gamma'_j = (\det \gamma_j) \gamma_j^{-1}$ . There is an integer  $j'$ ,  $0 \leq j' \leq p$ , such that  $\gamma'_j \in \Gamma \gamma_{j'}$ . We apply the action of  $\gamma'_j$  to (4) and get

$$\Psi \left( \frac{F(\tau)}{F_{j'}(\tau)}, t(\gamma_{j'} \tau) \right) = 0.$$

In other words, we have

$$\Psi' \left( \frac{F_{j'}(\tau)}{F(\tau)}, t(\gamma_{j'} \tau) \right) = 0$$

From this equality and (4), we see that if we let  $R(x, y)$  be the resultant of  $\Psi(z, x)$  and  $\Psi'(z, y)$  with respect to the variable  $z$ , then  $R(t(\tau), t(\gamma_{j'} \tau)) = 0$ . In particular, the modular polynomial  $\Phi_p(x, y)$  appears as an irreducible factor of  $R(x, y)$  over  $\mathbb{C}$ . This proves the lemma.  $\square$

We remark that, in general, there will be more than one irreducible factors in the numerator of the resultant  $R(x, y)$ , but it is not difficult to determine which one corresponds to  $\Phi_p(x, y)$ . For example, we can use the fact that the roots of  $\Phi_p(x, x)$  are the coordinates of certain CM-points (also known as *singular moduli*) to test which factor of  $R(x, y)$  is  $\Phi_p(x, y)$ . Thus, from the above discussion, we see that the most critical part of the calculation is the determination of the rational function  $\Psi(z, x)$ , which we address now.

From Newton's identity, we know that the problem of determining the symmetric sums of  $F_j(\tau)/F(\tau)$  in (3) is equivalent to that of determining  $(F_j/F)^m$  for  $m = 1, \dots, p+1$ . Observe that the sum of  $F_j^m$  is equal to  $p^{1-km/2} T_p F$ , by the definition of the Hecke operator  $T_p$ . Thus, to determine  $(F_j/F)^m$ , one just needs to know how the Hecke operator  $T_p$  acts on the basis given in Proposition 2. This is where the work of [15] comes into play.

In [15], we developed a method to compute  $T_{p_0}$ ,  $p_0$  a prime not dividing  $D$ , for arbitrary weight  $k$  with respect to the basis given in Proposition 2, assuming an explicit cover  $X_0^D(p_0)/W \rightarrow X_0^D(1)/W$  is known. Now recall the following explicit version of the Jacquet-Langlands correspondence.

**Proposition 4** ([6, 8]). *Let  $D$  be discriminant of an indefinite quaternion algebra over  $\mathbb{Q}$ . Let  $N$  be a positive integer relatively prime to  $D$ . For an Eichler order  $\mathcal{O} = \mathcal{O}(D, N)$  of level  $(D, N)$  and a positive even integer, let  $S_k(\Gamma(\mathcal{O}))$  denote the space of automorphic forms on the Shimura curve  $X_0^D(N)$ . Then*

$$S_k(\Gamma(\mathcal{O})) \simeq S_k^{D\text{-new}}(DN) := \bigoplus_{d|N} \bigoplus_{m|N/d} S_k^{\text{new}}(\Gamma_0(dD))^{[m]}$$

as Hecke modules. Here

$$S_k^{\text{new}}(\Gamma_0(dD))^{[m]} = \{f(m\tau) : f(\tau) \in S_k^{\text{new}}(\Gamma_0(dD))\}$$

and  $S_k^{\text{new}}(\Gamma_0(dD))$  denotes the newform subspace of cusp forms of weight  $k$  on  $\Gamma_0(dD)$ . In other words, for each Hecke eigenform  $f(\tau)$  in  $S_k^{D\text{-new}}(\Gamma_0(DN))$ , there corresponds a Hecke eigenform  $\tilde{f}(\tau)$  in  $S_k(\Gamma(\mathcal{O}))$  that shares the same Hecke eigenvalues. Moreover, for a prime divisor  $p$  of  $D$ , if the Atkin-Lehner involution  $W_p$  acts on  $f$  by  $W_p f = \epsilon_p f$ , then

$$W_p \tilde{f} = -\epsilon_p \tilde{f}.$$

For the situation under our consideration, we have

$$(5) \quad S_k^{\text{new}}(\Gamma_0(D), W) \simeq S_k(\Gamma),$$

where the left-hand side denotes the subspace of the newform subspace of cusp forms of weight  $k$  on  $\Gamma_0(D)$  that has eigenvalue  $-1$  for all  $w_q \in W$ ,  $q$  primes, and the right-hand

side denotes the space of automorphic forms of weight  $k$  on  $X$ . Thus, assuming the Hecke operator  $T_{p_0}$  on the space  $S_k^{\text{new}}(\Gamma_0(D), W)$  has no repeated eigenvalues, one can obtain the matrices for  $T_p$  with respect to the bases in Proposition 2 from those for  $T_{p_0}$  and the Fourier coefficients of Hecke eigenforms in  $S_k^{\text{new}}(\Gamma_0(D), W)$ .

In summary, to compute the modular equation  $\Phi_p(x, y)$ , we follow the following steps, assuming that the Schwarzian differential equation associated to  $X = X_0^D(1)/W$  and an explicit cover  $X_0^D(p_0)/W \rightarrow X_0^D(1)/W$  are known for some prime  $p_0$  not dividing  $D$ .

- (a) Pick a nonzero automorphic form  $F$  on  $X$  of the smallest possible weight  $k$ , expressed in the form given in Proposition 2.
- (b) Compute the matrices for  $T_{p_0}$  with respect to the basis in Proposition 2 for weights  $k, 2k, \dots, (p+1)k$  using the method in [15].
- (c) Compute Fourier coefficients of Hecke eigenforms in the space  $S_k^{\text{new}}(\Gamma_0(D), W)$  in (5) to the precision of  $p$  terms (using MAGMA [2] or SAGE [10]).
- (d) Compute the matrices for  $T_p$  with respect to the basis in Proposition 2 using informations from Steps (b) and (c). This gives us the expressions of  $\sum_{j=0}^p (F_j/F)^m$  in terms of  $t(\tau)$ , where  $F_j$  are defined by (2).
- (e) Use Newton's identity to convert expressions for  $\sum_{j=0}^p (F_j/F)^m$  to those for symmetric sums of  $F_j/F$  and hence determine the rational function  $\Psi(z, x)$  in (3).
- (f) Set  $\Psi'(z, y) = z^{p+1}\Psi(1/z, y)$ . Compute and factorize the resultant  $R(x, y)$  of  $\Psi(z, x)$  and  $\Psi'(z, y)$  with respect to the variable  $z$ .
- (h) Determine which irreducible factor of the numerator of  $R(x, y)$  is the modular polynomial  $\Psi_p(x, y)$  by using the fact that the roots of  $\Psi_p(x, x)$  are coordinates of some CM-points (also known as singular moduli) on  $X$ .

We now work out an example in details.

#### 4. AN EXAMPLE

In this section, we will work out the modular equation of level 7 for the Shimura curve  $X = X_0^{10}(1)/W_{10}$ . (Note that this case was not covered in [4].)

The curve  $X$  has 4 elliptic points of orders 2, 2, 2, 3, which we denote by  $P_2, P_2', P_2'', P_3$ , respectively. These are CM-points of discriminants  $-8, -20, -40$ , and  $-3$ , respectively. According to [4], there is a Hauptmodul  $t(\tau)$  that takes values  $\infty, 2$ , and  $27$ , and  $0$ , respectively. Using the covering  $X_0^{10}(3)/W_{10} \rightarrow X_0^{10}(1)/W_{10}$ , we [15, Equation (9)] found that the Schwarzian differential equation associated to  $t(\tau)$  is

$$\frac{d^2}{dt^2}F + \frac{3t^4 - 119t^3 + 3157t^2 - 7296t + 10368}{16t^2(t-2)^2(t-27)^2}F = 0.$$

Thus, by Proposition 1, near the point  $P_3$ , the  $t$ -expansion of  $t'(\tau)$  is the square of a linear combination of two solutions

$$F_1(t) = t^{1/3} \left( 1 - \frac{10}{81}t - \frac{18539}{839808}t^2 - \frac{168605}{25509168}t^3 - \frac{107269219465}{46548313473024}t^4 + \dots \right),$$

$$F_2(t) = t^{2/3} \left( 1 - \frac{5}{81}t - \frac{99095}{5878656}t^2 - \frac{8353325}{1428513408}t^3 - \frac{851170821485}{385081502367744}t^4 + \dots \right)$$

of the differential equation above. Thus, by Proposition 2, a basis for the space of automorphic forms of even weight  $k$  on  $X$  is

(6)

$$f_{k,j} = \frac{t^{j-1} (F_1(t) - CF_2(t))^{k/2}}{t^{\lfloor k/3 \rfloor} (1 - t/2)^{\lfloor k/4 \rfloor} (1 - t/27)^{\lfloor k/4 \rfloor}}, \quad j = 1, \dots, d_k = 1 - k + 3 \left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{k}{3} \right\rfloor,$$

for some complex number  $C$ . In particular, the one-dimensional space of automorphic forms of weight 4 on  $X$  is spanned by

$$F(\tau) = f_{4,1}(t(\tau)).$$

Let  $F_j(\tau)$ ,  $j = 0, \dots, 7$ , be defined by (2) with  $p = 7$ . According to the recipe described in Section 3, we first need to compute the matrices for  $T_3$  with respect to the basis in (6) for weights  $4m$ ,  $m = 1, \dots, 8$ . This has already been done in [15, Appendix D]. We found that the matrices of the Hecke operators  $T_3$  are

$k$	$A_k$
4	-8
8	28
12	$\begin{pmatrix} 468 & -98 \\ -1728 & 136 \end{pmatrix}$
16	$\begin{pmatrix} 1728 & 490 \\ 34560 & -3572 \end{pmatrix}$
20	$\begin{pmatrix} -2268 & -2450 \\ -328320 & 35992 \end{pmatrix}$
24	$\begin{pmatrix} 227772 & -272244 & 14406 \\ -388800 & -258192 & 12250 \\ 2985984 & 711936 & -199556 \end{pmatrix}$
28	$\begin{pmatrix} 420552 & 949620 & -72030 \\ -933120 & 4479732 & -61250 \\ -104509440 & 31147200 & -196568 \end{pmatrix}$
32	$\begin{pmatrix} 29821932 & -5456052 & 360150 \\ 95084928 & -48253536 & 306250 \\ 1803534336 & -618444288 & 19290988 \end{pmatrix}$

That is, for the integer  $k$  in the table, we have

$$T_3 \begin{pmatrix} f_{k,1} \\ \vdots \\ f_{k,d_k} \end{pmatrix} = A_k \begin{pmatrix} f_{k,1} \\ \vdots \\ f_{k,d_k} \end{pmatrix}.$$



The next informations we need are the Fourier expansions of Hecke eigenforms in the space  $S_k^{\text{new}}(\Gamma_0(10), -1, -1)$  for  $k = 4, 8, \dots, 32$ . By MAGMA [2], they are

$$\begin{aligned}
& q + 2q^2 - 8q^3 + \dots - 4q^7 + \dots, \\
& q + 2^3q^2 + 28q^3 + \dots + 104q^7 + \dots, \\
& q + 2^5q^2 + aq^3 + \dots + (-177a + 60500)q^7 + \dots, \\
& q + 2^7q^2 + aq^3 + \dots + (423a - 102460)q^7 + \dots, \\
& q + 2^9q^2 + aq^3 + \dots + (-417a + 48562100)q^7 + \dots, \\
& q + 2^{11}q^2 + aq^3 + \dots + \frac{1}{24}(-a^2 + 156628a + 145233723936)q^7 + \dots, \\
& q + 2^{13}q^2 + aq^3 + \dots + \frac{1}{84}(11a^2 - 17897672a - 75168751256976)q^7 + \dots, \\
& q + 2^{15}q^2 + aq^3 + \dots + \frac{1}{216}(a^2 - 14315428a + 407319502919904)q^7 + \dots,
\end{aligned}$$

respectively. Here each  $a$  is a root of the characteristic polynomial of  $T_3$  for the corresponding weight, and is different at each occurrence. From these, we deduce that the matrices for the Hecke operator  $T_7$  with respect to the bases in (6) are

$k$	$B_k$
4	-4
8	104
12	$\begin{pmatrix} -22336 & 17346 \\ 305856 & 36428 \end{pmatrix}$
16	$\begin{pmatrix} 628484 & 207270 \\ 14618880 & -1613416 \end{pmatrix}$
20	$\begin{pmatrix} 49507856 & 1021650 \\ 136909440 & 33553436 \end{pmatrix}$
24	$\begin{pmatrix} -826476664 & -2549118572 & 216037178 \\ -4554273600 & -3184965196 & 546964950 \\ 27509870592 & 52096359168 & 934073672 \end{pmatrix}$
28	$\begin{pmatrix} -91564144564 & 113245670860 & 5617780910 \\ 438283664640 & 412736094176 & -12502504350 \\ 15396166471680 & -2162558865600 & -111965170324 \end{pmatrix}$
32	$\begin{pmatrix} 4631981436536 & -203996300396 & 50284263050 \\ -11858411062656 & 12584751782372 & 97180354950 \\ 18304356630528 & 78355740427776 & 4460404162424 \end{pmatrix}$

Noticing that  $F^m = f_{4m, d_{4m}}$  for any positive integer  $m$ , we read from the matrices above that

$$\sum_{j=0}^7 \frac{F_j}{F} = -\frac{4}{7}, \quad \sum_{j=0}^7 \frac{F_j^2}{F^2} = \frac{104}{7^3}, \quad \sum_{j=0}^7 \frac{F_j^3}{F^3} = \frac{1}{7^5}(305856t^{-1} + 36428),$$

$$\begin{aligned}
\sum_{j=0}^7 \frac{F_j^4}{F^4} &= \frac{1}{7^7} (14618880t^{-1} - 1613416), & \sum_{j=0}^7 \frac{F_j^5}{F^5} &= \frac{1}{7^9} (136909440t^{-1} + 33553436), \\
\sum_{j=0}^7 \frac{F_j^6}{F^6} &= \frac{1}{7^{11}} (27509870592t^{-2} + 52096359168t^{-1} + 934073672), \\
\sum_{j=0}^7 \frac{F_j^7}{F^7} &= \frac{1}{7^{13}} (15396166471680t^{-2} - 2162558865600t^{-1} - 111965170324), \\
\sum_{j=0}^7 \frac{F_j^8}{F^8} &= \frac{1}{7^{15}} (18304356630528t^{-2} + 78355740427776t^{-1} + 4460404162424).
\end{aligned}$$

Then the rational function  $\Psi(z, x)$  in (3) is equal to

$$\begin{aligned}
\Psi(z, x) &= \frac{1}{7^{14}x^2} \Big( 678223072849x^2z^8 + 387556041628x^2z^7 + 7909306972x^2z^6 \\
&\quad - (527663765132x^2 + 4114130940864x)z^5 \\
&\quad + (46199115214x^2 - 5360751039168x)z^4 \\
&\quad + (72916497220x^2 - 2228082208128x)z^3 \\
&\quad + (90698975500x^2 - 75419213184x + 10905601867776)z^2 \\
&\quad + (-72866748500x^2 + 2516798571840x + 9093300682752)z \\
&\quad + (13624725625x^2 - 487484222400x + 10905601867776) \Big).
\end{aligned}$$

We then set  $\Psi'(z, y) = z^8\Psi(1/z, y)$  and compute the resultant  $R(x, y)$  of  $\Psi(z, x)$  and  $\Psi'(z, y)$  with respect to the variable  $z$ . The numerator of  $R(x, y)$  has two irreducible factors. To determine which one corresponds to the modular equation  $\Phi_7(x, y)$  of level 7, we use the fact that the roots of  $\Phi_7(x, x)$  should be the coordinates of CM-points of discriminants  $-3, -20, -27, -35, -40, -52, -115, -180$ , and  $-280$ , and these coordinates are all rational numbers. In fact, the coordinates of these CM-points were given in Table 3 of [4]. Those obtained numerically in [4] were later verified by Errthum [5] using Borchers forms. (In general, if  $\Phi_p(x, y)$  is the modular equation of level  $p$  for  $X$ , then the zeros of the polynomial  $\Phi_p(x, x)$  should be coordinates of CM-points of discriminants of the form  $(s^2 - 4p)/f^2$ ,  $(4s^2 - 8p)/f^2$ ,  $(25s^2 - 20p)/f^2$ , and  $(100s^2 - 40p)/f^2$ , subject to the condition that optimal embeddings of imaginary quadratic order of given discriminant into the maximal order in the quaternion algebra over  $\mathbb{Q}$  of discriminant 10 exist.) That is,  $\Phi_7(x, x)$  should factor into a product of linear factors over  $\mathbb{Q}$ . Indeed, exactly one of the two irreducible factors has this property. This determines  $\Phi_7(x, x)$ . The equation of  $\Phi_7(x, x)$  is given in Section 6.

## 5. MODULAR EQUATIONS FOR $X_0^6(1)/W_6$

In this section, we consider the Shimura curve  $X_0^6(1)/W_6$ . The Hauptmodul  $t$  is chosen such that it takes values 0, 1, and  $\infty$  at the CM-points of discriminant  $-24, -4$ , and  $-3$ , respectively. For a prime  $p \neq 2, 3$ , we let  $\Phi_p(x, y)$  denote the modular equation of level  $p$  for the Shimura curve  $X_0^6(1)/W_6$ . We have computed  $\Phi_p(x, y)$  for primes up to 19, but because the coefficients are very big, here we only list the equation for  $p = 7$ . (The equation for  $p = 5$ , with a slight change of variables, is contained in Appendix A of [15].) Files containing equations of other levels are available upon request.

We note that for  $p = 7$ , an explicit cover  $X_0^6(7)/W_6 \rightarrow X_0^6(1)/W_6$  have already been determined by Elkies [4]. It is easier to use this explicit cover to obtain the modular equation.

Write  $\Phi_7(x, y)$  as  $a_8(x)y^8 + \cdots + a_0(x)$ . Then

$$a_0(x) = (262254607552729x^2 - 121636570723920x + 501956755356672)^2,$$

$$\begin{aligned} a_1(x) = & -20948043194072943880152567521879704213891529442411744000000x^8 \\ & - 972224700022233770983527054177244292949817108052713267466888x^7 \\ & + 11273421679418251606098370957430742480477581392613807611381824x^6 \\ & + 1668909333131109923039277853641492658428121008149001087888384x^5 \\ & - 63984186761131298882488446421883340458529413111323886398291968x^4 \\ & + 44207311496458731717072123962700099989994411322119128098799616x^3 \\ & + 16376343297097580660696811581414332932110848991772493800800256x^2 \\ & - 10266396414223579734319217958394551183170348131151337958146048x \\ & - 61535122440667017884769009139405588503624662039298456944640, \end{aligned}$$

$$\begin{aligned} a_2(x) = & 38652996866188686573666243770903500386081949440000000000000x^8 \\ & - 754695394662453117523773206823864267539586547407817536000000x^7 \\ & + 106920403569574588553159546946652660456141317611471702546206108x^6 \\ & - 304712233218468979235116193697598666589103401841722665615811904x^5 \\ & + 760829927783454753461831780276599490081935198148332681198238208x^4 \\ & - 838656978040090256476182485622824884772400584504768719373500416x^3 \\ & + 279281854044007197025412087758954247401641295020911801816973312x^2 \\ & + 16376343297097580660696811581414332932110848991772493800800256x \\ & + 155040070253203796103748431879063330519757479108093010771968, \end{aligned}$$

$$\begin{aligned} a_3(x) = & -38038344753469448835190136766681303367680000000000000000000x^8 \\ & + 5356342063300144660882808541232703156528549197824000000000000x^7 \\ & + 763213772189169497838400901116835589760515183229150656000000000x^6 \\ & - 474977723264104825683894637831760410975632622470023372974366136x^5 \\ & + 167509988385063268988566748828192523079606899212802760635355200x^4 \\ & + 929561753522872554487774286564540945788659043162410938085107712x^3 \\ & - 838656978040090256476182485622824884772400584504768719373500416x^2 \\ & + 44207311496458731717072123962700099989994411322119128098799616x \\ & - 100063182095188848153455102416032183737981837866201188925440, \end{aligned}$$

We next give a short list of irrational singular moduli obtained by factorizing  $\Phi_p(x, x)$  for  $p \leq 19$ . Note that the norms of these coordinates were already computed in [5]. Our computation yields their exact values, not just their norms. (Note that all the rational

singular moduli for  $X_0^6(1)/W_6$  were numerically determined in [4] and later verified in [5].)

$d$	coordinates	norms
-264	$26198073x^2 - 53148528x + 42719296$	$2^6 19^2 43^2 / 3^9 11^3$
-276	$1771561x^2 + 328736070x - 206586207$	$3^8 23^1 37^2 / 11^6$
-136	$8703679193x^2 + 9087757328x + 52236012608$	$2^6 13^4 41^2 / 11^6 17^2$
-195	$1048576000000x^2 - 1812440448000x + 884218628241$	$3^8 13^2 19^2 47^2 / 2^{26} 5^6$
-420	$377149515625x^2 - 1357411709250x + 3732357388761$	$3^8 23^2 61^2 / 5^6 17^4$
-456	$42761175875209x^2 - 43342656704496x + 31050765799488$	$2^6 3^9 19^1 67^2 / 11^6 17^4$
-219	$1320903770112x^2 + 125096383466496x - 128030514913249$	$13^4 23^2 41^2 71^2 / 2^{26} 3^9$

## 6. MODULAR EQUATIONS FOR $X_0^{10}(1)/W_{10}$

In this section, we consider the Shimura curve  $X_0^{10}(1)/W_{10}$ . The Hauptmodul  $t$  is chosen such that it takes values 0, 2, 27, and  $\infty$  at the CM-points of discriminant  $-3$ ,  $-20$ ,  $-40$ , and  $-8$ , respectively. For a prime  $p \neq 2, 5$ , we let  $\Phi_p(x, y)$  denote the modular equation of level  $p$  for the Shimura curve  $X_0^{10}(1)/W_{10}$ . We have computed  $\Phi_p(x, y)$  for primes up to 23, but here we only list the equations for  $p = 7$ . Files containing equations of other levels are available upon request. (The equation for  $p = 3$  is given in Section 5 of [15].)

Write  $\Phi_7(x, y)$  as  $a_8(x)y^8 + \cdots + a_0(x)$ . Then

$$a_0(x) = x^2(278055625x^2 - 9948657600x + 222563303424)^3,$$

$$\begin{aligned} a_1(x) = & -22595841233117014160156250x^8 \\ & + 28984697923550260459599062500x^7 \\ & - 1645302326393970545433073200000x^6 \\ & + 6642716791333154856889291776000x^5 \\ & + 602202478597261111037649172561920x^4 \\ & - 4910763943297233363215784691630080x^3 \\ & - 8496599535482361705366127525232640x^2 \\ & + 9192477793078594137923003873230848x, \end{aligned}$$

$$\begin{aligned} a_2(x) = & 227107188088738387105484375x^8 \\ & - 10394755051722304542380018750x^7 \\ & + 2778503946540202482367606147300x^6 \\ & - 87342791194115920501573603009536x^5 \\ & + 509955665442187448278395883376640x^4 \\ & + 4112275508763979970115841213071360x^3 \\ & - 2548787139523595922308471728373760x^2 \\ & - 8496599535482361705366127525232640x \\ & + 11024545045545306955620945204609024, \end{aligned}$$

$$\begin{aligned}
a_3(x) = & -150467239701697875855937500x^8 \\
& - 66608231302181029579976003090x^7 \\
& + 1779457212256262244783949440310x^6 \\
& + 2537754854567828839953906813980x^5 \\
& - 567333242539975544914354048540992x^4 \\
& - 239926105649803444132294072811520x^3 \\
& + 4112275508763979970115841213071360x^2 \\
& - 4910763943297233363215784691630080x \\
& - 1478403072292996522939287129292800,
\end{aligned}$$

$$\begin{aligned}
a_4(x) = & 591743143857926286378268151x^8 \\
& + 3202866389133529516632265220x^7 \\
& - 354063672667325366031194372695x^6 \\
& + 22584679924916438377343486492770x^5 \\
& + 91353706850814722447426682881030x^4 \\
& - 567333242539975544914354048540992x^3 \\
& + 509955665442187448278395883376640x^2 \\
& + 602202478597261111037649172561920x \\
& + 107405110735755767009079459840000,
\end{aligned}$$

$$\begin{aligned}
a_5(x) = & -178678323923978351074342650x^8 \\
& + 71993226806139685483910592x^7 \\
& - 382859191369766872635900582760x^6 \\
& - 3892236690880311638995825770460x^5 \\
& + 22584679924916438377343486492770x^4 \\
& + 2537754854567828839953906813980x^3 \\
& - 87342791194115920501573603009536x^2 \\
& + 6642716791333154856889291776000x \\
& - 4678710992156906294833152000000,
\end{aligned}$$

$$\begin{aligned}
a_6(x) = & 38853800265463031036085625x^8 \\
& + 2535839489102333226932059850x^7 \\
& + 55519696363179781200215763218x^6 \\
& - 382859191369766872635900582760x^5 \\
& - 354063672667325366031194372695x^4 \\
& + 1779457212256262244783949440310x^3 \\
& + 2778503946540202482367606147300x^2 \\
& - 1645302326393970545433073200000x \\
& + 134184722882776668446400000000,
\end{aligned}$$

$$\begin{aligned}
a_7(x) = & -3904463975228746224750000x^8 \\
& - 279343739152597734767051250x^7 \\
& + 2535839489102333226932059850x^6 \\
& + 71993226806139685483910592x^5 \\
& + 3202866389133529516632265220x^4 \\
& - 66608231302181029579976003090x^3 \\
& - 10394755051722304542380018750x^2 \\
& + 28984697923550260459599062500x \\
& - 2307539315546608933125000000, \\
a_8(x) = & (528806915000x^4 - 3691767131325x^3 + 23850535042201x^2 \\
& - 2436693984375x + 4636577546875)^2.
\end{aligned}$$

Here we give a short table of irrational singular moduli obtained by factorizing  $\Phi_p(x, x)$  for  $p \leq 23$ . Note that the norms of these singular moduli were already computed in [5] using Borcherds forms. (All the rational singular moduli for  $X_0^{10}(1)/W_{10}$  were numerically determined in [4] and later verified in [5].)

$d$	coordinates	norms
-68	$25x^2 - 164x + 500$	$2^2 5^1$
-260	$31213x^2 - 7252x + 19652$	$2^2 17^3 / 7^4 13^1$
-360	$3844x^2 + 16299x + 132651$	$3^3 17^3 / 2^2 31^2$
-152	$1250x^3 - 62475x^2 + 64048x - 166375$	$11^3 / 2^1 5^1$
-132	$4225x^2 - 27108x + 364500$	$2^2 3^6 5^1 / 13^2$
-168	$122500x^2 - 647325x + 970299$	$3^6 11^3 / 2^2 5^4 7^2$
-195	$142129x^2 + 12489984x - 2985984$	$2^{12} 3^6 / 13^2 29^2$
-155	$74431x^2 - 1204224x - 5451776$	$2^{12} 11^3 / 7^4 31^1$
-440	$114244x^3 - 13732264x^2 + 20581253x - 16194277$	$11^3 23^3 / 2^2 13^4$
-83	$105625x^3 - 2894400x^2 + 30224384x + 32768000$	$2^{18} / 5^1 13^2$
-420	$67081x^2 + 431676x + 71118324$	$2^2 3^6 29^3 / 7^2 37^2$
-580	$13766909x^2 - 22395636x + 200973636$	$2^2 3^6 41^3 / 13^2 29^1 53^2$
-315	$72812089x^2 + 88682944x + 398688256$	$2^{15} 23^1 / 7^2 53^2$
-820	$3158777209x^2 - 19708111836x - 598885164$	$2^2 3^6 59^3 / 7^4 13^4 37^2$
-355	$253887551x^2 - 6128728704x + 3974344704$	$2^{12} 3^6 11^3 / 31^2 61^2 71$
-660	$4726150009x^2 - 10741876236x + 8174192436$	$2^2 3^9 47^3 / 7^4 23^2 61^2$
-680	$995781136x^3 - 8234027816x^2 + 166339878257x - 450688409963$	$11^3 17^3 41^3 / 2^4 7^6 23^2$
-435	$4460503369x^2 - 298408090176x + 25350000869376$	$2^{18} 3^9 17^3 / 7^4 29^2 47^2$
-920	$287478606136x^5 - 3389457456860x^4 - 4652502434310x^3$ $+ 216562501495x^2 + 50275785584380x - 60944362239977$	$23^2 59^3 / 2^3 29^1 47^2$

## REFERENCES

- [1] Pilar Bayer and Artur Travesa. Uniformizing functions for certain Shimura curves, in the case  $D = 6$ . *Acta Arith.*, 126(4):315–339, 2007.
- [2] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).
- [3] Reinier Bröker, Kristin Lauter, and Andrew V. Sutherland. Modular polynomials via isogeny volcanoes. *Math. Comp.*, 81(278):1201–1231, 2012.
- [4] Noam D. Elkies. Shimura curve computations. In *Algorithmic number theory (Portland, OR, 1998)*, volume 1423 of *Lecture Notes in Comput. Sci.*, pages 1–47. Springer, Berlin, 1998.
- [5] Eric Errthum. Singular moduli of Shimura curves. *Canad. J. Math.*, 63(4):826–861, 2011.
- [6] Hervé Jacquet and Robert P. Langlands. *Automorphic forms on  $GL(2)$* . Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin, 1970.

- [7] Santiago Molina. Ribet bimodules and the specialization of Heegner points. *Israel J. Math.*, 2012.
- [8] Hideo Shimizu. Theta series and automorphic forms on  $GL_2$ . *J. Math. Soc. Japan*, 24:638–683, 1972.
- [9] Goro Shimura. Construction of class fields and zeta functions of algebraic curves. *Ann. of Math. (2)*, 85:58–159, 1967.
- [10] W.A. Stein et al. *Sage Mathematics Software*. The Sage Development Team, 2012. <http://www.sagemath.org>.
- [11] Peter Stiller. Special values of Dirichlet series, monodromy, and the periods of automorphic forms. *Mem. Amer. Math. Soc.*, 49(299):iv+116, 1984.
- [12] John Voight. Quadratic forms and quaternion algebras: Algorithms and arithmetics. *Ph.D. thesis, University of California, Berkeley*, 2005.
- [13] John Voight and John Willis. Computing power series expansions of modular forms. *preprint*, 2012.
- [14] Yifan Yang. On differential equations satisfied by modular forms. *Math. Z.*, 246(1-2):1–19, 2004.
- [15] Yifan Yang. Schwarzian differential equations and Hecke eigenforms on Shimura curves. *Compositio Math.*, to appear, arxiv:1110.6284, 2012.

DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL CHIAO TUNG UNIVERSITY AND NATIONAL CENTER FOR THEORETICAL SCIENCES, HSINCHU, TAIWAN 300  
*E-mail address:* yfyang@math.nctu.edu.tw